THE EMPTINESS PROBLEM FOR AUTOMATA ON INFINITE TREES

Robert Hossley
Charles Rackoff

MAC Technical Memorandum 29

Spring 1972

This research was supported by the Advanced Research Projects Agency of the Department of Defense under ARPA Order No. 2095, and was monitored by ONR under Contract No. N00014-70-A-0362-0006

Massachusetts Institute of Technology

PROJECT MAC

Cambridge Massachusetts 02139

19

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
PROJECT MAC

2a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

2b. GROUP
NONE

3. REPORT TITLE
THE EMPTINESS PROBLEM FOR AUTOMATA ON INFINITE TREES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
INTERIM SCIENTIFIC REPORT

5. AUTHOR(S) *(First name, middle initial, last name)*
ROBERT HOSSLEY AND CHARLES RACKOFF

6. REPORT DATE
AUGUST 1972

7a. TOTAL NO. OF PAGES
16

7b. NO. OF REFS
6

8a. CONTRACT OR GRANT NO.
N00014-70-A-0362-0006

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)
MAC-TM-29

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
NONE

10. DISTRIBUTION STATEMENT
DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY
OFFICE OF NAVAL RESEARCH

13. ABSTRACT

The purpose of ~~this~~ paper is to give an alternative proof to the decidability of the emptiness problem for tree automata, as shown in Rabin [4]. The proof reduces the emptiness problem for automata on infinite trees to that for automata on finite trees, by showing that any automata definable set of infinite trees must contain a finitely-generable tree.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Infinite trees						
S2S (monadic second-order theory of two successors)						
Infinite games						
Tree automata						
Emptiness problem						

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The purpose of this paper is to give an alternative proof to the decidability of the emptiness problem for tree automata, as shown in Rabin [4]. The proof reduces the emptiness problem for automata on infinite trees to that for automata on finite trees, by showing that any automata definable set of infinite trees must contain a finitely-generable tree.

Section 1: Introduction

The analysis of finite automata on infinite trees is the basis for Rabin's remarkable proof of the decidability of S2S (the monadic second-order theory of two successors) [5]. Rabin's proof follows the now standard form of Buchi and Elgot's proof for WS1S (weak, single successor) [1, 3] and Thatcher-Wright's proof for weak S2S, and requires demonstrating effectively that the automata are closed under union, projection, and negation, and that the emptiness problem for the automata is decidable. As in the case of S1S, the main technical difficulty in the case of S2S lies in proving closure under complementation of sets accepted by nondeterministic automata on infinite trees. The problem is complicated by the fact that nondeterministic infinite tree automata are known not to be equivalent to any of the likely definitions of deterministic infinite tree automata.

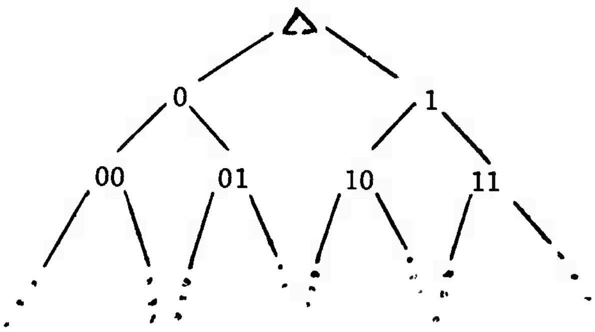
Curiously, the emptiness problem, which is easy for the other kinds of automata, turns out to be nontrivial for (nondeterministic) infinite tree automata. Rabin subsequently improved his original proof of the decidability of this emptiness problem, but even the second proof [4] used an involved induction and consequently does not yield a simple effective criterion for deciding emptiness.

In this paper we provide such a criterion by showing that an infinite tree automaton accepts some valued tree if and only if there is a computation of the automaton containing a certain simple kind of finite subtree. Moreover, the set of finite subtrees of the kind we require are recognizable by *finite* tree automata, and in this way we reduce the emptiness problem for infinite tree automata directly to that for finite tree automata. This also yields a simpler proof of another result of Rabin about "regular" runs by automata (see below).

The hardest part of Rabin's proof -- the complementation lemma -- remains a difficult combinatorial argument which has yet to be simplified. Reducing this problem to the corresponding problem of complementing finite-tree automata (which is easily resolved by the usual subset construction) might lead to such a simplification. Our results on emptiness suggest that there is hope for this approach.

Section 2:

For this paper the appropriate way to visualize the infinite binary tree T is as follows. At the top is the root Λ. Every $x \in T$ has a left son x0 and a right son x1. Hence, $T = \{0,1\}^*$.



We define a partial ordering on T by $x \le y$ (x is an *initial* of y) if $y = xz$ for some $z \in \{0,1\}^*$. If $x \le y$ and $x \ne y$, then we will write $x < y$.

A *path* π of T is a set $\pi \subset T$ satisfying 1) $\Lambda \in \pi$; 2) for $y \in \pi$, either $y0 \in \pi$ or $y1 \in \pi$, but not both; 3) π is a minimal subset of T satisfying 1 and 2.

Definition: If for some path π, $x \in \pi$ and $y \in \pi$, then we denote by $[x,y]$ the *set* $\{w | x \le w \le y\}$. Note that when $y < x$, $[x,y] = \emptyset$.

For a set B we denote the cardinality of B by c(B).

Definition: A set $B \subset T$ is called a *frontier* of T if for every path $\pi \subset T$ we have $c(\pi \cap B) = 1$. By König's Lemma every frontier is finite. A *finite tree* is a set $E = \{y | y \le w$, for some $w \in B\}$, where B is a fixed frontier of T. For E as above, B is called the frontier of E and is denoted by Ft(E).

Definition: A *Σ-tree* is a pair (v,T) such that $v: T \to \Sigma$. A *finite Σ-tree* is a pair (v,E) such that $v: E \to \Sigma$, where E is a finite tree.

Definition: For a mapping $\theta: A \to B$. $\text{In}(\theta) = \{b | c(\theta^{-1}(b)) \ge \omega\}$.

Definition: Let $\theta: A \to B$ and let $\Omega = ((L_i, U_i))_{1 \le i \le n}$ be a sequence of pairs of finite sets. We say that *θ is of type Ω*, denoted by $\theta \in [\Omega]$, if for *some* i, $1 \le i \le n$, we have $\text{In}(\theta) \cap L_i = \emptyset$ and $\text{In}(\theta) \cap U_i \neq \emptyset$.

Definition: An *f.a.t.* is a system $\mathfrak{A} = \langle S, \Sigma, M, s_0, \Omega \rangle$, where S is a finite set of *states*, Σ is a finite set, $M: S \times \Sigma \to P(S \times S)$, $s_0 \in S$ is the *initial state*, and $\Omega = ((L_i, U_i))_{1 \le i \le n}$.

If $t = (v,T)$ is a Σ-tree, then an *$\mathfrak{A}$-run on t* is any mapping $r: T \to S$ such that:

1) $r(\Lambda) = s_0$, and
2) for all $y \in T$,

$$(r(y0),\ r(y1)) \in M(r(y),\ v(y)).$$

If $e = (v,E)$ is a finite Σ-tree, then an *$\mathfrak{A}$-run on e* is any mapping $r: E \to S$ such that:

1) $r(\Lambda) = s_0$, and
2) for all $y \in E - Ft(E)$,

$$(r(y0),\ r(y1)) \in M(r(y),\ v(y)).$$

The set of all $\mathfrak{A}$-runs on $t(e)$ will be denoted by $Rn(\mathfrak{A},t)(Rn(\mathfrak{A},e)$, respectively). An *accepting $\mathfrak{A}$-run* on t is any $r \in Rn(\mathfrak{A},t)$ such that for every path $\pi \subset T$, $(r|\pi) \in [\Omega]$. $T(\mathfrak{A}) = \{t \mid$ there is an accepting $\mathfrak{A}$-run on $t\}$. $T(\mathfrak{A})$ is called the set defined by $\mathfrak{A}$.

Given an f.a.t. $\mathfrak{A} = < S,\ \Sigma,\ M,\ s_0, \Omega>$ we wish to determine whether or not $T(\mathfrak{A}) = \phi$. Consider the automaton $\bar{\mathfrak{A}} = < S,\ \{a\},\ \bar{M},\ s_0, \Omega>$, where for all $s \in S$, $\bar{M}(s,a) = \bigcup_{\sigma \in \Sigma} M(s,\sigma)$. Clearly, $T(\mathfrak{A}) = \phi$ iff $T(\bar{\mathfrak{A}}) = \phi$.

Thus the emptiness problem is reduced to the case of automata over the single letter alphabet $\{a\}$. Henceforth we restrict our attention to this case. Since there exists just one $\{a\}$-tree, $(\bar{v},T)$, and for every finite tree E just one finite $\{a\}$-tree, $(\bar{v},E\ )$, we will omit mention of the valuation $\bar{v}$ and talk about $\mathfrak{A}$-runs on T and E, $\mathfrak{A}$ accepting T, etc.

Theorem 1: Let $\mathfrak{A} = < S, \{a\}, M, s_0, ((L_i, U_i))_{1 \le i \le n} >$ be an f.a.t. $T(\mathfrak{A}) \ne \emptyset \Leftrightarrow$ for some finite tree E there exists an r such that

1) $r \in Rn(\mathfrak{A}, E)$,

2) there exist mappings $J: Ft(E) \rightarrow E\text{-}Ft(E)$ and $H: Ft(E) \rightarrow E\text{-}Ft(E)$ such that for all $x \in Ft(E)$

a) $H(x) \le J(x) < x$,

b) $r(J(x)) = r(x)$,

c) $r([H(x), J(x)]) = r([J(x), x])$,

d) for some i, $1 \le i \le n$, $r([J(x), x]) \cap L_i = \emptyset$ and $r(x) \in U_i$.

Before we prove Theorem 1, we show that Theorem 1 easily yields the following theorem.

Theorem 2: The emptiness problem for f.a.t.'s is decidable.

Proof of Theorem 2: Let $\mathfrak{A}$ be as in the statement of Theorem 1.

Definition: Let E be a tree (finite or infinite). Let r be an $\mathfrak{A}$-run on E. Let $x \in E$. Since $x \in \{0,1\}^*$ we can write $x = \sigma_1\sigma_2 \ldots \sigma_m$. Define $\alpha_{r,x}$ to be the following member of S^*: $\alpha_{r,x} = r(\Lambda) \cdot r(\sigma_1) \cdot r(\sigma_1\sigma_2) \ldots r(x)$.

Notation: Let α be a string. Let n and m be positive integers, $n \le m$. Then by $\alpha(n)$ we will mean the nth element (from the left) of α. By $\alpha([n,m])$ we will mean the set of elements between and including the nth and the mth places of α.

Definition: Let $\alpha \in S^*$. We say that α is Good if there exist positive integers H and J such that $H \leq J < N = \text{length}(\alpha)$, $\alpha(J) = \alpha(N)$, $\alpha([H,J]) = \alpha([J,N])$, and there exists an i such that $\alpha(N) \in U_i$ and $\alpha([J,N]) \cap L_i = \phi$. Note that good is defined with respect to our f.a.t. $\mathfrak{A}$.

Lemma 1: The set of good strings is a regular set, i.e., it is recognizable by a finite state machine on finite input strings.

Proof of Lemma 1: Obvious. □

Lemma 2: Let G be a regular set of finite strings on S. Let H = $\{E|E$ is a finite tree and there exists a run r on E such that for all $x \in Ft(E)$, $\alpha_{r,x} \in G\}$ Then H is recognizable by a finite automaton on finite trees as defined in [6].

Proof of Lemma 2: Fairly obvious. □

Completion of proof of Theorem 2: By Theorem 1, Lemma 1, and Lemma 2, the emptiness problem for $\mathfrak{A}$ can be reduced to the emptiness problem for a particular finite automaton on finite trees. But by Theorem 7 in [6], this problem is decidable.

Proof of ⇒ in Theorem 1: Let r be an accepting $\mathfrak{A}$-run on T. By the definition of accepting run and of good string, it is clear that for every path π of T there exists an x, $x \in \pi$, such that $\alpha_{r,x}$ is a good string. Let $B = \{x|\alpha_{r,x}$ is good and for all $y < x$, $\alpha_{r,y}$ isn't good$\}$. Then B is a frontier. If we let E be the finite tree with frontier B, then there exist mappings J and H which, together with $r|E$, satisfy conditions 1 and 2 of Theorem 1. This completes the proof of ⇒.

Proof of $\Leftarrow$ in Theorem 1: Let E, r, J and H be as specified in 1) and 2) of Theorem 1.

We define a mapping η: $T \to E$ inductively as follows. Let $\eta(\Delta) = \Delta$. If $\eta(x)$ has been defined, then for $\sigma \in \{0,1\}$, define $\eta(x\sigma)$ as follows.

Case 1: If $\eta(x) \in E - Ft(E)$, then let $\eta(x\sigma) = \eta(x)\cdot\sigma$.

Case 2: If $\eta(x) \in Ft(E)$, then let $\eta(x\sigma) = J(\eta(x))\cdot\sigma$.

Define $\bar{r}$: $T \to S$ by $\bar{r}(x) = r(\eta(x))$, for all $x \in T$. Clearly by 2) b) of Theorem 1, $\bar{r} \in Rn(\mathfrak{A}, T)$ so that it suffices to show that for all paths $\pi \subset T$, $(\bar{r} \restriction \pi) \in [\Omega]$, because then $T \in \mathbb{T}(\mathfrak{A})$ and hence $T(\mathfrak{A}) \neq \emptyset$.

Let $\pi \subset T$ be a specific path. Let $y_0, y_1, y_2, \ldots$ be the infinite subset of π (listed in increasing order under $\leq$) consisting of exactly those members of π whose images under η are in $Ft(E)$. Define V_π to be the following infinite sequence of members of $Ft(E) \times Ft(E)$:

$$V_\pi = (\eta(y_0), \eta(y_1)), (\eta(y_1), \eta(y_2)), (\eta(y_2), \eta(y_3)), \ldots$$

For all $i < \omega$ we have by the definition of η, $J(\eta(y_i)) < \eta(y_{i+1})$ and $\bar{r}([y_i, y_{i+1}]) = r([J(\eta(y_i)), \eta(y_{i+1})])$. Hence, $In(\bar{r} \mid \pi) =$

$$\bigcup_{(x,z) \in In(V_\pi)} r([J(x), z]).$$

Clearly there exists a finite sequence (possibly with repetition) of members of $Ft(E)$, $x_1, x_2, x_3, \ldots, x_m$, such that

$$\text{(I)} \qquad \begin{aligned} & x_1 = x_m \text{ and} \\ & In(V_\pi) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), \ldots, \\ & (x_{m-1}, x_m)\}. \end{aligned}$$

From now on we will denote $J(x_i)$ by J_i and $H(x_i)$ by H_i, for all $1 \le i \le m$.

We have from the preceding paragraph

$$\text{(II)} \qquad \begin{aligned} & \text{for all } 1 \le i < m,\ J_i < x_{i+1}, \\ & \text{and } In(\bar{r} \mid \pi) = \bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]). \end{aligned}$$

$(\bar{r} \mid \pi) \in [\Omega]$ is immediate from the third of the following three lemmas.

<u>Lemma 3</u>: There exists an M, $1 \le M \le m$, such that for all i, $1 \le i \le m$,

$$H_M \le H_i.$$

That is, $H_M = \underline{\min}\{H_1, \ldots, H_m\}$.

<u>Proof</u>: Our induction hypothesis at stage h is that there exists an integer M', $1 \le M' \le h$, such that for all i, $1 \le i \le h$, $H_{M'} \le H_i$. Clearly the basis case is trivial. We assume the induction hypothesis for h and prove it for h+1.

$H_{M'} \le H_h$, by the induction hypothesis.

$H_h \le J_h$, by 2) a) in Theorem 1.

$J_h < x_{h+1}$, by (II).

Hence, $H_{M'} < x_{h+1}$.

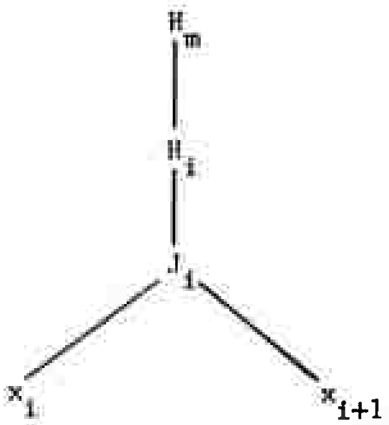
By 2) a) of Theorem 1 we also have $H_{h+1} < x_{h+1}$. Therefore, H_M and H_{h+1} are comparable (under ≤). Clearly for all i, $1 \le i \le h+1$, $\underline{\min}\{H_{M'}, H_{h+1}\} \le H_i$. □

If $M \ne m$, we can rename $x_1, x_2, \ldots, x_m$ so that (I) and (II) remain true and $H_m = \underline{\min}\{H_1, \ldots, H_m\}$. Henceforth, without loss of generality we assume that $M = m$.

Lemma 4: If $H_m = \underline{\min}\{H_1, \ldots, H_m\}$, then for all i, $1 \le i \le (m-1)$,

$$r([H_m, x_{i+1}]) \supseteq r([H_m, x_i]).$$

Proof: Let i be any integer such that $1 \le i < m$. $H_m \le H_i \le J_i < x_{i+1}$, hence we have the picture:

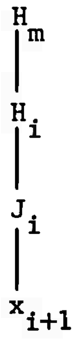


Hence, $r([H_m, x_{i+1}]) \supseteq r([H_m, H_i]) \cup r([H_i, J_i])$. By 2) c) of Theorem 1, $r([H_i, J_i]) = r([J_i, x_i])$. Hence, $r([H_m, x_{i+1}]) \supseteq r([J_i, x_i])$, and therefore, $r([H_m, x_{i+1}]) \supseteq r([H_m, x_i])$. $\square$

Lemma 5: If $H_m = \underline{\min}\{H_1, \ldots, H_m\}$, then for all i, $1 \le i \le (m-1)$, $r([H_m, x_m]) \supseteq r([J_i, x_{i+1}])$.

Proof: Let i be any integer such that $1 \le i \le (m-1)$.

By Lemma 4 $r([H_m, x_m]) \supseteq r([H_m, x_{m-1}])$, $r([H_m, x_{m-1}]) \supseteq r([H_m, x_{m-2}])$, $\ldots$, $r([H_m, x_{i+2}]) \supseteq r([H_m, x_{i+1}])$. Hence, $r([H_m, x_m]) \supseteq r([H_m, x_{i+1}])$. We have $H_m \le H_i \le J_i < x_{i+1}$. That is the picture:



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Hence, $[H_m, x_{i+1}] \supseteq [J_i, x_{i+1}]$. Hence $r([H_m, x_m]) \supseteq r([J_i, x_{i+1}])$. $\square$

Completion of the Proof of Theorem 1: Without loss of generality we assume $H_m = \underline{\min}\{H_1, \ldots, H_m\}$. By Lemma 5,

$$r([H_m, x_m]) \supseteq \bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]).$$

By part 2) d) of Theorem 1 we have for some i, $1 \le i \le n$, $r([J_m, x_m] \cap L_i = \emptyset$ and $r(x_m) \in U_i$. By part 2) c) of Theorem 1, $r([H_m, x_m]) = r([J_m, x_m])$. Hence,

$$\bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]) \cap L_i = \emptyset,$$

and

$$\bigcup_{i=1}^{m-1} r([J_i, x_{i+1}]) \cap U_i \neq \emptyset.$$

Therefore, by (IJ) $(\bar{r} \mid \pi) \in [\Omega]$.

$\square$

Section 3: Remarks

In [5] Rabin uses the following definition.

Definition: An *f.a.t. with designated subsets* is a system $\mathfrak{A} = \langle S, \Sigma, M, s_0, \mathcal{F} \rangle$, where S is a finite set of *states*, Σ is a finite set, $M\colon S \times \Sigma \to P(S \times S)$, and $\mathcal{F} \subseteq P(S)$ is the set of *designated subsets*. An $\mathfrak{A}$-run on $t = (v,T)$ is as defined in Section 2. $\mathfrak{A}$ *accepts* t if there exists an $r \in Rn(\mathfrak{A},t)$ such that for all paths $\pi \subset T$, $In(r \mid \pi) \in \mathcal{F}$.

The proof of Theorem 1 can be extended to show that $r([H_m, x_m]) = \bigcup_{i=1}^{m-1} r([J_i, x_{i+1}])$, where H_i, x_i, etc. are as in the proof of Theorem 1.

Hence for $\mathfrak{A} = \langle S, \{a\}, M, s_0, \mathcal{F} \rangle$, where $c(S) = q$, we have: $T(\mathfrak{A}) \neq \emptyset \Leftrightarrow$ for some finite tree E there exists an r such that

1) $r \in Rn(\mathfrak{A},E)$,

2) there exist mappings $J\colon Ft(E) \to E\text{-}Ft(E)$ and $H\colon Ft(E) \to E\text{-}Ft(E)$ such that

   a) $H(x) \leq J(x) < x$,

   b) $r(J(x)) = r(x)$,

   c) $r([H(x), J(x)]) = r([J(x), x])$,

   d) $r([J(x), x]) \in \mathcal{F}$.

The appropriate definition of a good string with respect to $\mathfrak{A}$ is a simple modification of the definition of good string used in the proof of Theorem 2. For either definition of good string we can design a non-deterministic finite automaton on finite strings, $\mathfrak{M}$, which recognizes the set of good strings and which has at most $2^{2q}(q+1)$ states. By the subset construction we can design a deterministic automaton $\mathfrak{M}'$ equivalent to $\mathfrak{M}$ such that $\mathfrak{M}'$ has at most $Q = 2^{2^{2q}(q+1)}$ states. Using $\mathfrak{M}'$ we can easily construct a finite automaton on finite trees, $\mathfrak{A}'$, such that $T(\mathfrak{A}') \neq \emptyset$ if and only if $T(\mathfrak{A}) \neq \emptyset$ and such that the state set of $\mathfrak{A}'$ is the cross product of the state sets of $\mathfrak{A}$ and $\mathfrak{M}'$. Hence $\mathfrak{A}'$ has at most qQ states. We can determine whether $T(\mathfrak{A}') \neq \emptyset$ in $(q\ Q)^3$ computational steps.

Hence given a finite automaton $\mathfrak{A}$ on infinite trees which has q states and uses either notion of acceptance, we can determine whether or not $T(\mathfrak{A}) \neq \emptyset$ in $\left(q\ 2^{2^{2q}(q+1)}\right)^3$ computational steps.

Remark 2: If we have a finite Σ-tree (v,E), and a function $J\colon Ft(E) \to E-Ft(E)$ such that for all $x \in Ft(E)$, $v(J(x)) = v(x)$, then we can generate a unique Σ-tree $(\bar{v},T)$ as in the proof of Theorem 1. Call any Σ-tree which can be generated in this way a *finitely-generable* Σ-tree.

Rabin in [4] defines a Σ-tree, (v,T), to be regular if and only if for each $\sigma \in \Sigma$, $v^{-1}(\sigma)$ is a regular subset of $\{0,1\}^*$. It is easily shown that a Σ-tree is finitely-generable if and only if it is regular.

Remark 3: From Theorem 1 it is easily shown that if an f.a.t. accepts any Σ-tree, then it accepts a finitely-generable Σ-tree. Rabin shows this in [4]. In [2] Buchi and Landweber prove that if $P(X,Y)$ is a finite-state condition and X has a winning strategy, then X has a winning finite-state strategy. Rabin and Rackoff have independently observed that the set of winning strategies for X corresponds in a natural way to a set of $\{0,1\}$-trees defined by a (deterministic) infinite tree automaton. Hence, it *easily* follows from Rabin's result in [4] or from the results in this paper that if X has a winning strategy then X has a winning finite-state strategy.

C. Rackoff has observed the following. If X does *not* have a winning strategy, then by our Theorem 1 we see that X does not have a "*partial*" strategy of a particular kind. From this one can show that Y has a winning strategy for $P(X,Y)$, thus showing that $P(X,Y)$ is determined. This is another result of [2].

We thank Albert Meyer for his assistance in writing this paper and for pointing out the relevance of finite tree automata to our work.

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